

Gröbner-Shirshov bases for Lie Ω -algebras and free Rota-Baxter Lie algebras*

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Abstract: In this paper, we generalize the Lyndon-Shirshov words to Lyndon-Shirshov Ω -words on a set X and prove that the set of all non-associative Lyndon-Shirshov Ω -words forms a linear basis of the free Lie Ω -algebra on the set X . From this, we establish Gröbner-Shirshov bases theory for Lie Ω -algebras. As applications, we give Gröbner-Shirshov bases for free λ -Rota-Baxter Lie algebras, free modified λ -Rota-Baxter Lie algebras and free Nijenhuis Lie algebras and then linear bases of such three free algebras are obtained.

Key words: Lie Ω -algebra; Nijenhuis Lie algebra; Rota-Baxter Lie algebra; Lyndon-Shirshov word; Gröbner-Shirshov basis.

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1 Introduction

Let (R, \cdot) be an algebra over a field k , $\lambda \in k$ and $P : R \rightarrow R$ a linear map satisfying

$$P(x) \cdot P(y) = P(P(x) \cdot y) + P(x \cdot P(y)) + \lambda P^n(x \cdot y), \quad x, y \in R,$$

where n is a nonnegative integer. If $n = 1$ (resp. $n = 0$, $n = 2$ and $\lambda = -1$), then the operator P is called a λ -Rota-Baxter (resp. modified λ -Rota-Baxter, Nijenhuis) operator on the algebra (R, \cdot) and the triple (R, \cdot, P) is called a λ -Rota-Baxter (resp. modified λ -Rota-Baxter, Nijenhuis) algebra.

Rota-Baxter operator on an associative algebra was introduced by G. Baxter to solve an analytic problem in probability [6] and was studied by G.-C. Rota [51] in combinatorics later. The Rota-Baxter operator of weight $\lambda = 0$ on Lie algebra is also called the operator form of the classical Yang-Baxter equation

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due to Semenov-Tian-Shansky's work [52]. Modified Rota-Baxter operator on an associative algebra was introduced by K. Ebrahimi-Fard [33] with motivation from modified classical Yang-Baxter equation on a Lie algebra [52]. The Nijenhuis operator on an associative algebra was introduced by J. Cariñena et al. [24] to study quantum bi-Hamiltonian systems. In [56], Nijenhuis operators are constructed by analogy with Poisson-Nijenhuis geometry, from relative Rota-Baxter operators. Nijenhuis operators on Lie algebras play an important role in the study of integrability of nonlinear evolution equations [28]. Recently, there are some results on Rota-Baxter operators on Lie algebras and related topic, for example, see [1, 4, 5, 47, 49].

There are many constructions of free λ -Rota-Baxter associative algebras, free modified λ -Rota-Baxter associative algebras and free Nijenhuis associative algebras by using different methods, for example, [3, 14, 25, 34–42, 46, 51]. However, as we know, there are no constructions of free λ -Rota-Baxter Lie algebras, free modified λ -Rota-Baxter Lie algebras and free Nijenhuis Lie algebras. We will apply Gröbner-Shirshov bases method to construct such three free Lie algebras.

Gröbner bases and Gröbner-Shirshov bases were invented independently by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras [54, 55], free Lie algebras [53, 55] and implicitly free associative algebras [53, 55] (see also [7, 9]), by H. Hironaka [44] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [21] for ideals of the polynomial algebras. Gröbner bases and Gröbner-Shirshov bases theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra. It is a powerful tool to solve the following classical problems: normal form; word problem; conjugacy problem; rewriting system; automaton; embedding theorem; PBW theorem; extension; homology; growth function; Dehn function; complexity; etc. See, for example, the books [2, 18, 22, 23, 27, 30], the papers [7–9, 13, 29, 37, 48], and the surveys [11, 15–17].

The concept of Ω -algebra was introduced by A.G. Kurosh [45] under an influence of the concept of multioperator group of P.J. Higgins [43]. An Ω -algebra over a field k is a k -algebra A with a set of multilinear operators Ω , where $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$ and each Ω_m is a set of m -ary multilinear operators on A . λ -Rota-Baxter Lie algebras, modified λ -Rota-Baxter Lie algebras and Nijenhuis Lie algebras are Lie Ω -algebras with one operator. In [14], the associative Ω -words on a set X are introduced, it is shown that the set of all associative Ω -words on X forms a linear basis of the free associative Ω -algebra on X , and then Gröbner-Shirshov bases theory for associative Ω -algebras is established. For Gröbner-Shirshov bases theory of various Ω -algebras and their applications, see [12, 14, 32, 37, 50].

The first linear basis of a free Lie algebra $Lie(X)$ on a set X had been given by M. Hall 1950 [31]. K.-T. Chen, R.H. Fox, R.C. Lyndon 1958 [26] and A.I. Shirshov 1958 [53] introduced non-associative Lyndon-Shirshov words on X and proved that they form a linear basis of $Lie(X)$, independently. One of the main applications of Lyndon-Shirshov basis is the Shirshov's theory of Gröbner-Shirshov bases theory for Lie algebras [53].

The paper is organized as follows. In section 2, we review the concept and some related properties of Lyndon-Shirshov words on a set X , generalize associative (resp. non-associative) Lyndon-Shirshov words to associative (resp. non-associative) Lyndon-Shirshov Ω -words on a set X and show that the set of all non-associative Lyndon-Shirshov Ω -words forms a linear basis of the free Lie Ω -algebra on the set X . In section 3, we review Gröbner-Shirshov bases theory for associative Ω -algebras, give definition of Gröbner-Shirshov bases for Lie Ω -algebras and establish Composition-Diamond lemma for Lie Ω -algebras. In section 4, we give Gröbner-Shirshov bases for free λ -Rota-Baxter Lie algebras, free modified λ -Rota-Baxter Lie algebras, free Nijenhuis Lie algebras and then linear bases of such three free algebras are obtained by Composition-Diamond lemma for Lie Ω -algebras.

2 Free Lie Ω -algebras

2.1 Lyndon-Shirshov words

In this subsection, we review the concept and some properties of Lyndon-Shirshov words, which can be found in [10, 53, 55].

For any set X , we define the following notations:

$S(X)$: the set of all nonempty associative words on X .

X^* : the set of all associative words on X including the empty word 1.

X^{**} : the set of all non-associative words on X .

For any $u \in X^*$, denote $\deg(u)$ to be the degree (length) of u . Let $>$ be a well order on X . Define the lex-order $>_{lex}$ and the deg-lex order $>_{deg-lex}$ on X^* with respect to $>$ by:

(i) $1 >_{lex} u$ for any nonempty word u , and if $u = x_i u'$ and $v = x_j v'$, where $x_i, x_j \in X$, then $u >_{lex} v$ if $x_i > x_j$, or $x_i = x_j$ and $u' >_{lex} v'$ by induction.

(ii) $u >_{deg-lex} v$ if $\deg(u) > \deg(v)$, or $\deg(u) = \deg(v)$ and $u >_{lex} v$.

A nonempty associative word w is called an associative Lyndon-Shirshov word on X , if $w = uv >_{lex} vu$ for any decomposition of $w = uv$, where $1 \neq u, v \in X^*$.

A non-associative word $(u) \in X^{**}$ is said to be a non-associative Lyndon-Shirshov word on X with respect to the lex-order $>_{lex}$, denoted by $[u]$, if

- (a) u is an associative Lyndon-Shirshov word on X ;
- (b) if $(u) = ((v)(w))$, then both (v) and (w) are non-associative Lyndon-Shirshov words on X ;
- (c) if $(v) = ((v_1)(v_2))$, then $v_2 \leq_{lex} w$.

Denote the set of all associative (resp. non-associative) Lyndon-Shirshov words on X with respect to the lex-order $>_{lex}$ by $ALSW(X)$ (resp. $NLSW(X)$). It is well known that for any $u \in ALSW(X)$, there exists a unique Shirshov standard bracketing way $[u]$ such that $[u] \in NLSW(X)$. Then $NLSW(X) = \{[u] | u \in ALSW(X)\}$.

Let $k\langle X \rangle$ be the free associative algebra on X over a field k and $Lie(X)$ be the Lie subalgebra of $k\langle X \rangle$ generated by X under the Lie bracket $(uv) = uv - vu$. It is well known that $Lie(X)$ is a free Lie algebra on the set X with a linear basis $NLSW(X)$.

For any $f \in k\langle X \rangle$, let \bar{f} be the leading word of f with respect to the deg-lex order $>_{deg-lex}$ on X^* .

Lemma 2.1 ([10, 53, 55]) *For any non-associative word $(u) \in X^{**}$, (u) has a representation*

$$(u) = \sum \alpha_i [u_i],$$

in $Lie(X)$, where each $\alpha_i \in k$ and $u_i \in ALSW(X)$.

Lemma 2.2 ([10, 53, 55]) *If $u \in ALSW(X)$, then $\overline{[u]} = u$.*

Lemma 2.3 ([10, 53, 55]) *For any $u \in X^*$, there exists a unique decomposition*

$$u = u_1 u_2 \cdots u_m,$$

where each $u_i \in ALSW(X)$ and $u_j \leq_{lex} u_{j+1}$, $1 \leq j \leq m-1$.

Lemma 2.4 ([10, 53, 55]) *Let $u = avb$, where $u, v \in ALSW(X)$ and $a, b \in X^*$. Then*

$$[u] = [a[v]d],$$

where $b = cd$ for some $c, d \in X^$. Let*

$$[u]_v = [u]_{[vc] \mapsto [\cdots [[v][c_1]][c_2]] \cdots [c_m]}$$

where $c = c_1 c_2 \cdots c_m$ with each $c_i \in ALSW(X)$ and $c_i \leq_{lex} c_{i+1}$. Then,

$$[u]_v = a[v]b + \sum \alpha_i a_i [v] b_i,$$

where each $\alpha_i \in k, a_i, b_i \in X^$ and $a_i v b_i <_{deg-lex} avb = u$. It follows that $\overline{[u]_v} = u$.*

2.2 Lyndon-Shirshov Ω -words

In this subsection, we define the Lyndon-Shirshov Ω -words on a set X .

Let

$$\Omega = \bigcup_{m=1}^{\infty} \Omega_m,$$

where Ω_m is a set of m -ary operators for any $m \geq 1$. For any set Y , denote

$$\Omega(Y) := \bigcup_{m=1}^{\infty} \left\{ \omega^{(m)}(y_1, y_2, \dots, y_m) \mid y_i \in Y, 1 \leq i \leq m, \omega^{(m)} \in \Omega_m \right\}.$$

Let X be a set. Define $\langle \Omega; X \rangle_0 := S(X)$ and $(\Omega; X)_0 := X^{**}$. Assume that we have defined $\langle \Omega; X \rangle_{n-1}$ and $(\Omega; X)_{n-1}$. Define

$$\begin{aligned}\langle \Omega; X \rangle_n &:= S(X \cup \Omega(\langle \Omega; X \rangle_{n-1})), \\ (\Omega; X)_n &:= (X \cup \Omega((\Omega; X)_{n-1}))^{**}.\end{aligned}$$

Then it is easy to see that for any $n \geq 0$,

$$\langle \Omega; X \rangle_n \subseteq \langle \Omega; X \rangle_{n+1}, \quad (\Omega; X)_n \subseteq (\Omega; X)_{n+1}.$$

Denote

$$\langle \Omega; X \rangle := \bigcup_{n=0}^{\infty} \langle \Omega; X \rangle_n, \quad (\Omega; X) := \bigcup_{n=0}^{\infty} (\Omega; X)_n.$$

The elements of $\langle \Omega; X \rangle$ (resp. $(\Omega; X)$) are called the associative (resp. non-associative) Ω -words on X .

If $u \in X \cup \Omega(\langle \Omega; X \rangle)$, then u is called prime. Therefore, for any $u \in \langle \Omega; X \rangle$, u can be expressed uniquely in the canonical form

$$u = u_1 u_2 \cdots u_n, \quad n \geq 1,$$

where each u_i is prime. The number n is called the breath of u , which is denoted by $bre(u)$. The degree of u , denoted by $deg(u)$, is defined to be the total number of all occurrences of all $x \in X$ and $\theta \in \Omega$ in u . For any associative Ω -word u , define the depth of u to be

$$dep(u) := \min\{n | u \in \langle \Omega; X \rangle_n\}.$$

Let (u) be a non-associative Ω -word. Define the depth of (u) by $dep((u)) = dep(u)$.

For example, if $u = \omega^{(3)}(x_2 x_1 x_1, x_1, \omega^{(1)}(x_2 x_2 x_1)) x_2 x_1$, where $x_1, x_2 \in X$ and $\omega^{(3)}, \omega^{(1)} \in \Omega$, then $deg(u) = 11$, $bre(u) = 3$ and $dep(u) = 2$.

Let $u = u_1 u_2 \cdots u_n, n \geq 1$, where each u_i is prime. Denote

$$wt(u) := (deg(u), bre(u), u_1, u_2, \dots, u_n).$$

Let X and Ω be well-ordered sets with the orders $>_X$ and $>_\Omega$, respectively. Define the Deg-lex order $>_{Dl}$ on $\langle \Omega; X \rangle$ as follows. For any $u = u_1 u_2 \cdots u_n, v = v_1 v_2 \cdots v_m$, where u_i, v_j are prime, define

$$u >_{Dl} v \text{ if } wt(u) > wt(v) \text{ lexicographically,}$$

where if $u_i = \omega(u_{i1}, u_{i2}, \dots, u_{it}), v_i = \theta(v_{i1}, v_{i2}, \dots, v_{il})$ and $deg(u_i) = deg(v_i)$, then $u_i > v_i$ if

$$(\omega, u_{i1}, u_{i2}, \dots, u_{it}) > (\theta, v_{i1}, v_{i2}, \dots, v_{il}) \text{ lexicographically.}$$

Let \succ be the restriction of $>_{Dl}$ on $X \cup \Omega(\langle \Omega; X \rangle)$. We define the Lyndon-Shirshov Ω -words on the set X by induction on the depth of the Ω -words.

For $n = 0$, define

$$ALSW(\Omega; X)_0 := ALSW(X),$$

$$NLSW(\Omega; X)_0 := NLSW(X) = \{[u] | u \in ALSW(\Omega; X)_0\}$$

with respect to the lex-order \succ_{lex} on X^* .

Assume that we have defined

$$ALSW(\Omega; X)_{n-1} \quad \text{and} \quad NLSW(\Omega; X)_{n-1} = \{[u] | u \in ALSW(\Omega; X)_{n-1}\}.$$

Define

$$ALSW(\Omega; X)_n := ALSW(X \cup \Omega(ALSW(\Omega; X)_{n-1}))$$

with respect to the lex-order \succ_{lex} .

For any $u \in X \cup \Omega(ALSW(\Omega; X)_{n-1})$, define the bracketing way on u by

$$[u] := \begin{cases} u, & \text{if } u \in X, \\ \omega^{(m)}([u_1], [u_2], \dots, [u_m]), & \text{if } u = \omega^{(m)}(u_1, u_2, \dots, u_m). \end{cases}$$

Denote

$$[X \cup \Omega(ALSW(\Omega; X)_{n-1})] := \{[u] | u \in X \cup \Omega(ALSW(\Omega; X)_{n-1})\}.$$

Then, the order \succ on $X \cup \Omega(ALSW(\Omega; X)_{n-1})$ induces an order (still denoted by \succ) on $[X \cup \Omega(ALSW(\Omega; X)_{n-1})]$ by $[u] \succ [v]$ if $u \succ v$ for any $u, v \in X \cup \Omega(ALSW(\Omega; X)_{n-1})$. If $u = u_1 u_2 \dots u_m \in ALSW(\Omega; X)_n$, where each $u_i \in X \cup \Omega(ALSW(\Omega; X)_{n-1})$, then we define

$$[u] := [[u_1][u_2] \dots [u_m]]$$

is the Shirshov standard bracketing way on $\{[u_1], [u_2], \dots, [u_m]\}$, which means $[u]$ a non-associative Lyndon-Shirshov word on $\{[u_1], [u_2], \dots, [u_m]\}$ with respect to the lex-order \succ_{lex} . Denote

$$NLSW(\Omega; X)_n := \{[u] | u \in ALSW(\Omega; X)_n\}.$$

It is easy to see that

$$NLSW(\Omega; X)_n = NLSW([X \cup \Omega(ALSW(\Omega; X)_{n-1})]).$$

Denote

$$\begin{aligned} ALSW(\Omega; X) &:= \bigcup_{n=0}^{\infty} ALSW(\Omega; X)_n, \\ NLSW(\Omega; X) &:= \bigcup_{n=0}^{\infty} NLSW(\Omega; X)_n. \end{aligned}$$

Then

$$NLSW(\Omega; X) = \{[u] | u \in ALSW(\Omega; X)\}.$$

The elements of $ALSW(\Omega; X)$ (resp. $NLSW(\Omega; X)$) are called the associative (resp. non-associative) Lyndon-Shirshov Ω -words. By the above definitions, for any associative Lyndon-Shirshov Ω -word u , u is associated a unique non-associative Lyndon-Shirshov Ω -word $[u]$. For example, if

$$u = \omega^{(3)}(x_2 x_1 x_1, x_1, \omega^{(1)}(x_2 x_2 x_1)) x_2 x_1,$$

where $x_1, x_2 \in X$ with $x_2 > x_1$, and $\omega^{(3)}, \omega^{(1)} \in \Omega$, then $u \in ALSW(\Omega; X)$ and

$$[u] = (\omega^{(3)}(((x_2 x_1) x_1), x_1, \omega^{(1)}((x_2(x_2 x_1))))(x_2 x_1)) \in NLSW(\Omega; X).$$

2.3 Free Lie Ω -algebras

In this subsection, we prove that the set $NLSW(\Omega; X)$ of all non-associative Lyndon-Shirshov Ω -words is a linear basis of the free Lie Ω -algebra on the set X .

An associative (resp. non-associative, Lie) Ω -algebra over a field k is an associative (resp. non-associative, Lie) algebra A with multilinear operators Ω .

For any associative Ω -algebra (A, \cdot, Ω) , it is easy to see that $(A, [,], \Omega)$ is a Lie Ω -algebra, where

$$[a, a'] = a \cdot a' - a' \cdot a, \quad a, a' \in A.$$

Let X be a set. An associative (resp. non-associative, Lie) Ω -algebra $F(X)$ together with a injective map $i : X \rightarrow F(X)$ is called a free associative (resp. non-associative, Lie) Ω -algebra on X , if for any associative (resp. non-associative, Lie) Ω -algebra A and any map $\sigma : X \rightarrow A$, there exists a unique associative (resp. non-associative, Lie) Ω -algebra homomorphism $\tilde{\sigma} : F(X) \rightarrow A$ such that $\tilde{\sigma}i = \sigma$.

Let $k\langle\Omega; X\rangle$ be the k -linear space spanned by $\langle\Omega; X\rangle$. Then $k\langle\Omega; X\rangle$ is a free associative Ω -algebra on the set X , see [14]. Denote $Lie(\Omega; X)$ the Lie Ω -subalgebra of $k\langle\Omega; X\rangle$ generated by X under the Lie bracket $(uv) = uv - vu$. The elements of $k\langle\Omega; X\rangle$ (resp. $Lie(\Omega; X)$) are called associative (resp. Lie) Ω -polynomials on X .

Lemma 2.5 *For any $(u) \in (\Omega; X)$, (u) has a representation*

$$(u) = \sum \alpha_i [u_i],$$

in $Lie(\Omega; X)$, where each $\alpha_i \in k$ and $[u_i] \in NLSW(\Omega; X)$.

Proof. Induction on $dep((u))$. If $dep((u)) = 0$, by Lemma 2.1, we have $(u) = \sum \alpha_i [u_i]$, where each $\alpha_i \in k$ and $[u_i] \in NLSW(X) \subseteq NLSW(\Omega; X)$.

Assume that the result is true for any (u) with $dep((u)) \leq n - 1$.

Let $dep((u)) = n \geq 1$. There are two cases to consider.

Case 1. If $(u) = \theta((u_1), (u_2), \dots, (u_m))$, then $\text{dep}((u_i)) \leq n-1, 1 \leq i \leq m$. Thus, by induction, we may assume that $(u_i) = [u_i]$, where each $[u_i] \in \text{NLSW}(\Omega; X)$. Therefore,

$$(u) = \theta([u_1], [u_2], \dots, [u_m]) = [\theta(u_1, u_2, \dots, u_m)] \in \text{NLSW}(\Omega; X).$$

Case 2. If $(u) = (a_1 a_2 \dots a_m), m \geq 2$, where each a_i is prime, then by Case 1, we may assume that $(u) = ([a_1][a_2] \dots [a_m])$, where each $a_i \in X \cup \Omega(\text{ALSW}(\Omega; X)_{n-1})$. Thus, (u) is a non-associative word on $\{[a_1], [a_2], \dots, [a_m]\}$. By Lemma 2.1, we have

$$([a_1][a_2] \dots [a_m]) = \sum \alpha_i [u_i],$$

where each $\alpha_i \in k$ and $[u_i] \in \text{NLSW}(\{[a_1], [a_2], \dots, [a_m]\}) \subseteq \text{NLSW}(\Omega; X)$. \square

For any $f \in k\langle \Omega; X \rangle$, let \bar{f} be the leading Ω -word of f with respect to the order $>_{Dl}$ on $\langle \Omega; X \rangle$.

Lemma 2.6 *If $u \in \text{ALSW}(\Omega; X)$, then $\bar{u} = u$ with respect to the order $>_{Dl}$ on $\langle \Omega; X \rangle$.*

Proof. Induction on $\text{dep}(u)$. If $\text{dep}(u) = 0$, then $\bar{u} = u$ by Lemma 2.2. Assume that the result is true for any $u \in \text{ALSW}(\Omega; X)$ with $\text{dep}(u) \leq n-1$.

Let $\text{dep}(u) = n \geq 1$. There are two cases to consider.

Case 1. If $\text{bre}(u) = 1$ and $u = \theta(u_1, u_2, \dots, u_m)$, then $[u] = \theta([u_1], [u_2], \dots, [u_m])$. By induction, we have $\bar{[u_i]} = u_i, 1 \leq i \leq m$. Thus,

$$\bar{[u]} = \theta(\bar{[u_1]}, \bar{[u_2]}, \dots, \bar{[u_m]}) = \theta(u_1, u_2, \dots, u_m) = u.$$

Case 2. If $\text{bre}(u) > 1$ and $u = a_1 a_2 \dots a_m$, where each a_i is prime, then by Lemma 2.2, we have

$$[u] = [[a_1][a_2] \dots [a_m]] = [a_1][a_2] \dots [a_m] + \sum \alpha_i [a_{i_1}][a_{i_2}] \dots [a_{i_m}],$$

where each $\{i_1, i_2, \dots, i_m\} = \{1, 2, \dots, m\}$ and $a_{i_1} a_{i_2} \dots a_{i_m} <_{Dl} a_1 a_2 \dots a_m$. By Case 1, we have $\bar{[a_t]} = a_t$ and $\bar{[a_{i_j}]} = a_{i_j}$. It follows that $\bar{[u]} = a_1 a_2 \dots a_m = u$. \square

Lemma 2.7 *$\text{NLSW}(\Omega; X)$ is a linear basis of $\text{Lie}(\Omega; X)$.*

Proof. Suppose $\sum_{i=1}^m \alpha_i [u_i] = 0$ in $\text{Lie}(\Omega; X)$, where each $\alpha_i \in k, u_i \in \text{ALSW}(\Omega; X)$ and $u_i >_{Dl} u_{i+1}$. If $\alpha_1 \neq 0$, then by Lemma 2.6, we have $\sum_{i=1}^m \alpha_i \bar{[u_i]} = u_1$, a contradiction. Therefore, $\text{NLSW}(\Omega; X)$ is linear independent set. By Lemma 2.5, $\text{NLSW}(\Omega; X)$ is a linear basis of $\text{Lie}(\Omega; X)$. \square

Let $k(\Omega; X)$ be the k -linear space spanned by $(\Omega; X)$. It is easy to see that $k(\Omega; X)$ is a free non-associative Ω -algebra on the set X . Denote R_Ω the set consisting of the following relations in $k(\Omega; X)$:

$$((u)(v)) = -((v)(u)),$$

$$(((u)(v))(w)) = (((u)(w))(v)) + ((u)((v)(w))),$$

where $(u), (v), (w) \in (\Omega; X)$. Then $FL_\Omega(X) := k(\Omega; X)/Id(R_\Omega)$ is a free Lie Ω -algebra on the set X .

Theorem 2.8 *Lie($\Omega; X$) is a free Lie Ω -algebra on the set X with a linear basis $NLSW(\Omega; X)$.*

Proof. Let $i : X \rightarrow FL_\Omega(X), x \mapsto x + Id(R_\Omega)$ and $\varphi : X \rightarrow k\langle\Omega; X\rangle, x \mapsto x$. Since $FL_\Omega(X)$ is a free Lie Ω -algebra on X , there is a unique Lie Ω -algebra homomorphism $\tilde{\varphi} : FL_\Omega(X) \rightarrow k\langle\Omega; X\rangle$ such that $\tilde{\varphi}i = \varphi$. It is easy to see that

$$\tilde{\varphi}(FL_\Omega(X)) = Lie(\Omega; X).$$

Similar to proof of Lemma 2.5, we have for any $(u) \in (\Omega; X)$,

$$(u) + Id(R_\Omega) = \sum \beta_j [u_j] + Id(R_\Omega),$$

where each $\beta_j \in k$ and $u_j \in ALSW(\Omega; X)$. If

$$\tilde{\varphi}(\sum_{i=1}^m \alpha_i [u_i] + Id(R_\Omega)) = \sum_{i=1}^m \alpha_i [u_i] = 0$$

in $k\langle\Omega; X\rangle$, where each $\alpha_i \in k, u_i \in ALSW(\Omega; X)$, then by Lemma 2.7, we have each $\alpha_i = 0$. Thus, $\tilde{\varphi}$ is injective. It follows that $FL_\Omega(X) \cong Lie(\Omega; X)$, i.e., $Lie(\Omega; X)$ is a free Lie Ω -algebra on the set X . \square

3 Gröbner-Shirshov bases for Lie Ω -algebras

3.1 Composition-Diamond lemma for associative Ω -algebras

In this subsection, we review Gröbner-Shirshov bases theory for associative Ω -algebras, which can be found in [14].

Let $k\langle\Omega; X\rangle$ be the free associative Ω -algebra on X and $\star \notin X$. By a \star - Ω -word we mean any expression in $\langle\Omega; X \cup \{\star\}\rangle$ with only one occurrence of \star . The set of all \star - Ω -word on X is denoted by $\langle\Omega; X\rangle^\star$.

Let π be a \star - Ω -word and $s \in k\langle\Omega; X\rangle$. Then we call

$$\pi|_s := \pi|_{\star \mapsto s}$$

an s - Ω -word.

Now, we assume that $\langle \Omega; X \rangle$ is equipped with a monomial order $>$. This means that $>$ is a well order on $\langle \Omega; X \rangle$ such that for any $v, w \in \langle \Omega; X \rangle$ and $\pi \in \langle \Omega; X \rangle^*$, if $w > v$, then $\pi|_w > \pi|_v$.

For every Ω -polynomial $f \in k\langle \Omega; X \rangle$, let \bar{f} be the leading Ω -word of f with respect to the order $>$. If the coefficient of \bar{f} is 1, then we call that f is monic. We also call a set $S \subseteq k\langle \Omega; X \rangle$ monic if each $s \in S$ is monic.

Let $f, g \in k\langle \Omega; X \rangle$ be monic. Then we define two kinds of compositions.

- (I) If there exists an associative Ω -word $w = \bar{f}a = b\bar{g}$ for some $a, b \in \langle \Omega; X \rangle$ such that $bre(w) < bre(\bar{f}) + bre(\bar{g})$, then we call $(f, g)_w := fa - bg$ the intersection composition of f and g with respect to the ambiguity w .
- (II) If there exists an associative Ω -word $w = \bar{f} = \pi|_{\bar{g}}$ for some $\pi \in \langle \Omega; X \rangle^*$, then we call $(f, g)_w := f - \pi|_g$ the inclusion composition of f and g with respect to the ambiguity w .

Let $S \subseteq k\langle \Omega; X \rangle$ be monic. The composition $(f, g)_w$ is called trivial modulo (S, w) if

$$(f, g)_w = \sum \alpha_i \pi_i|_{s_i},$$

where each $\alpha_i \in k$, $\pi_i \in \langle \Omega; X \rangle^*$, $s_i \in S$ and $\pi_i|_{\bar{s}_i} < w$. If this is the case, we write

$$(f, g)_w \equiv_{ass} 0 \text{ mod}(S, w).$$

In general, for any two associative Ω -polynomials p and q , $p \equiv_{ass} q \text{ mod}(S, w)$ means that $p - q = \sum \alpha_i \pi_i|_{s_i}$, where each $\alpha_i \in k$, $\pi_i \in \langle \Omega; X \rangle^*$, $s_i \in S$ and $\pi_i|_{\bar{s}_i} < w$.

A monic set S is called a Gröbner-Shirshov basis in $k\langle \Omega; X \rangle$ if any composition $(f, g)_w$ of $f, g \in S$ is trivial modulo (S, w) .

Lemma 3.1 ([14], Composition-Diamond lemma for associative Ω -algebras)
Let $S \subseteq k\langle \Omega; X \rangle$ be monic, $>$ a monomial order on $\langle \Omega; X \rangle$ and $Id_{ass}(S)$ the ideal of $k\langle \Omega; X \rangle$ generated by S . Then the following statements are equivalent:

- (i) S is a Gröbner-Shirshov basis in $k\langle \Omega; X \rangle$.
- (ii) $f \in Id_{ass}(S) \Rightarrow \bar{f} = \pi|_{\bar{s}}$ for some $\pi \in \langle \Omega; X \rangle^*$ and $s \in S$.
- (iii) The set $Irr(S) = \{w \in \langle \Omega; X \rangle | w \neq \pi|_{\bar{s}}, \pi \in \langle \Omega; X \rangle^*, s \in S\}$ is a linear basis of the associative Ω -algebra $k\langle \Omega; X | S \rangle := k\langle \Omega; X \rangle / Id_{ass}(S)$.

3.2 Composition-Diamond lemma for Lie Ω -algebras

In this subsection, we establish Composition-Diamond lemma for Lie Ω -algebras, which is a generalization of the Shirshov's Composition-Diamond lemma for Lie algebras.

Let $>_{DI}$ be the order defined as before. It is easy to see that $>_{DI}$ is a monomial order on $\langle \Omega; X \rangle$. We always use this order in this subsection, in

particular, a Gröbner-Shirshov basis in $k\langle\Omega; X\rangle$ is with respect to the order $>_{DI}$.

The following lemma follows from Lemma 2.3.

Lemma 3.2 *Let $u = u_1 u_2 \cdots u_m$, where each $u_i \in X \cup \Omega(ALSW(\Omega; X))$. Then there exists a unique decomposition*

$$u = c_1 c_2 \cdots c_t,$$

where each $c_i \in ALSW(\Omega; X)$ and $c_j \preceq_{lex} c_{j+1}$, $1 \leq j \leq t-1$.

Lemma 3.3 *Let $\pi \in \langle\Omega; X\rangle^*$ and $v, \pi|_v \in ALSW(\Omega; X)$. Then there is a $\pi_1 \in \langle\Omega; X\rangle^*$ and $c \in \langle\Omega; X\rangle$ such that*

$$[\pi|_v] = [\pi_1|_{[vc]}],$$

where c may be empty. Let

$$[\pi|_v]_v := [\pi_1|_{[vc]}]_{[vc] \mapsto [\cdots [[v][c_1]][c_2] \cdots [c_m]]},$$

where $c = c_1 c_2 \cdots c_m$ with each $c_i \in ALSW(\Omega; X)$ and $c_t \preceq_{lex} c_{t+1}$, $1 \leq t \leq m-1$. Then,

$$[\pi|_v]_v = \pi|_{[v]} + \sum \alpha_i \pi_i|_{[v]},$$

where each $\alpha_i \in k$ and $\pi_i|_v <_{DI} \pi|_v$. It follows that $\overline{[\pi|_v]_v} = \pi|_v$ with respect to the order $>_{DI}$.

Proof. Induction on the depth of $\pi|_v$. If $dep(\pi|_v) = 0$, then the result is true by Lemma 2.4. Assume that the result is true for any $\pi|_v$ with $dep(\pi|_v) \leq n-1$.

Let $dep(\pi|_v) = n \geq 1$. There are two cases to consider.

Case 1. If $\pi|_v = avb$, where $a, b \in \langle\Omega; X\rangle$. Let $a = a_1 a_2 \cdots a_l$, $l \geq 0$ and $b = b_1 b_2 \cdots b_t$, $t \geq 0$, where a_i and b_j are prime. By Lemma 2.4, we can obtain

$$[\pi|_v] = [[a_1][a_2] \cdots [a_l][vc][b_{j+1}] \cdots [b_t]] = [a[vc]d],$$

where $c = b_1 b_2 \cdots b_j$ and $d = b_{j+1} b_{j+2} \cdots b_t$. Let $c = c_1 c_2 \cdots c_m$, where each $c_i \in ALSW(\Omega; X)$ and $c_1 \preceq_{lex} \cdots \preceq_{lex} c_m$. Then,

$$[\pi|_v]_v = [[a_1][a_2] \cdots [a_l][\cdots [[v][c_1]][c_2] \cdots [c_m]]][b_{j+1}] \cdots [b_t]]$$

and by Lemma 2.4, we have

$$[\pi|_v]_v = [a_1][a_2] \cdots [a_l][v][b_1][b_2] \cdots [b_t] + \sum \beta_i [d_{i_1}] \cdots [d_{i_p}][v][d_{i_{p+1}}] \cdots [d_{i_{p+t}}].$$

Therefore, by Lemma 2.6, we have

$$[\pi|_v]_v = a[v]b + \sum \alpha_j a'_j[v]b'_j,$$

where each $a'_j v b'_j <_{DI} avb$.

Case 2. If $\pi|_v = a\theta(u_1, u_2, \dots, \pi'|_v, \dots, u_q)b$, where $a, b \in \langle \Omega; X \rangle$ and may be empty, then we have

$$[\pi|_v] = [a\theta(u_1, u_2, \dots, [\pi'|_v], \dots, u_q)b].$$

By induction,

$$[\pi'|_v] = [\pi''|_{[vc]}], \quad [\pi'|_v]_v = \pi'|_{[v]} + \sum \alpha_i \pi'_i|_{[v]},$$

where $\pi'_i|_v <_{DI} \pi'|_v$. Therefore,

$$[\pi|_v] = [a\theta(u_1, u_2, \dots, [\pi''|_{[vc]}], \dots, u_q)b]$$

and

$$\begin{aligned} [\pi|_v]_v &= [a\theta(u_1, \dots, \pi'|_v, \dots, u_q)b]_v \\ &= [a\theta([u_1], \dots, [\pi'|_v]_v, \dots, [u_q])b] \\ &= a\theta([u_1], \dots, [\pi'|_v]_v, \dots, [u_q])b + \sum \alpha_i a_i \theta([u_1], \dots, [\pi'|_v]_v, \dots, [u_q])b_i \\ &= a\theta(u_1, \dots, \pi'|_{[v]}, \dots, u_q)b + \sum \beta_j a'_j \theta(u'_{j_1}, u'_{j_2}, \dots, \pi''_j|_{[v]}, \dots, u'_{j_q})b'_j \\ &= \pi|_{[v]} + \sum \beta_j a'_j \theta(u'_{j_1}, u'_{j_2}, \dots, \pi''_j|_{[v]}, \dots, u'_{j_q})b'_j, \end{aligned}$$

where each $a'_j \theta(u'_{j_1}, \dots, \pi''_j|_v, \dots, u'_{j_q})b'_j <_{DI} \pi|_v$. It follows that $\overline{[\pi|_v]}_v = \pi|_v$. \square

Definition 3.4 Let $\pi \in \langle \Omega; X \rangle^*$ and $f \in Lie(\Omega; X) \subseteq k\langle \Omega; X \rangle$ be monic. If $\pi|_{\bar{f}} \in ALSW(\Omega; X)$, then

$$[\pi|_f]_{\bar{f}} := [\pi|_{\bar{f}}]_{\bar{f}}|_{[\bar{f}] \mapsto f}$$

is called a special normal f -word.

Corollary 3.5 Let $f \in Lie(\Omega; X)$ and $\pi|_{\bar{f}} \in ALSW(\Omega; X)$. Then

$$[\pi|_f]_{\bar{f}} = \pi|_f + \sum \alpha_i \pi_i|_f,$$

where each $\alpha_i \in k$ and $\pi_i|_{\bar{f}} <_{DI} \pi|_{\bar{f}}$.

Let $f, g \in Lie(\Omega; X)$ be monic. There are two kinds of compositions.

- (a) If $w = \bar{f}a = b\bar{g}$, where $a, b \in \langle \Omega; X \rangle$ with $bre(w) < bre(\bar{f}) + bre(\bar{g})$, then

$$\langle f, g \rangle_w := [fa]_{\bar{f}} - [bg]_{\bar{g}}$$

is called the intersection composition of f and g with respect to the ambiguity w .

(b) If $w = \bar{f} = \pi|_{\bar{g}}$, then

$$\langle f, g \rangle_w := f - [\pi|_g]_{\bar{g}}$$

is called the inclusion composition of f and g with respect to the ambiguity w .

If S is a monic subset of $Lie(\Omega; X)$, then the composition $\langle f, g \rangle_w$ is called trivial modulo (S, w) if

$$\langle f, g \rangle_w = \sum \alpha_i [\pi_i|_{s_i}]_{\bar{s}_i},$$

where each $\alpha_i \in k$, $s_i \in S$, $[\pi_i|_{s_i}]_{\bar{s}_i}$ is a special normal s_i -word and $\pi_i|_{\bar{s}_i} <_{Dl} w$. If this is the case, then we write

$$\langle f, g \rangle_w \equiv 0 \mod(S, w).$$

In general, for any two Lie Ω -polynomials p and q , $p \equiv q \mod(S, w)$ means that $p - q = \sum \alpha_i [\pi_i|_{s_i}]_{\bar{s}_i}$, where each $\alpha_i \in k$, $s_i \in S$, $[\pi_i|_{s_i}]_{\bar{s}_i}$ is a special normal s_i -word and $\pi_i|_{\bar{s}_i} <_{Dl} w$.

Definition 3.6 A monic set S is called a Gröbner-Shirshov basis in $Lie(\Omega; X)$ if any composition $\langle f, g \rangle_w$ of $f, g \in S$ is trivial modulo (S, w) .

Lemma 3.7 Let $f, g \in Lie(\Omega; X) \subset k\langle \Omega; X \rangle$ be monic. Then

$$\langle f, g \rangle_w - (f, g)_w \equiv_{ass} 0 \mod(\{f, g\}, w).$$

Proof. If $\langle f, g \rangle_w$ and $(f, g)_w$ are compositions of intersection, where $w = \bar{f}a = b\bar{g}$, then

$$\langle f, g \rangle_w = [fa]_{\bar{f}} - [bg]_{\bar{g}} = fa + \sum \alpha_i a_i f a'_i - bg - \sum \beta_j b_j g b'_j,$$

where $a_i \bar{f} a'_i, b_j \bar{g} b'_j <_{Dl} w$. It follows that

$$\langle f, g \rangle_w - (f, g)_w \equiv_{ass} 0 \mod(\{f, g\}, w).$$

If $\langle f, g \rangle_w$ and $(f, g)_w$ are compositions of inclusion, where $w = \bar{f} = \pi|_{\bar{g}}$, then

$$\langle f, g \rangle_w = f - [\pi|_g]_{\bar{g}} = f - \pi|_g - \sum \alpha_i \pi_i|_g,$$

where $\pi_i|_{\bar{g}} <_{Dl} w$. It follows that

$$\langle f, g \rangle_w - (f, g)_w \equiv_{ass} 0 \mod(\{f, g\}, w). \quad \square$$

Lemma 3.8 Let $S \subset Lie(\Omega; X) \subset k\langle \Omega; X \rangle$ be monic. With the order $>_{Dl}$ on $\langle \Omega; X \rangle$, the following two statements are equivalent:

- (i) S is a Gröbner-Shirshov basis in $Lie(\Omega; X)$.
- (ii) S is a Gröbner-Shirshov basis in $k\langle \Omega; X \rangle$.

Proof. (i) \Rightarrow (ii). Suppose that S is a Gröbner-Shirshov basis in $Lie(\Omega; X)$. Then, for any composition $\langle f, g \rangle_w$, we have $\langle f, g \rangle_w = \sum \alpha_i [\pi_i|_{s_i}]_{\overline{s_i}}$, where each $\alpha_i \in k, \pi_i|_{\overline{s_i}} <_{Dl} w, \pi_i \in \langle \Omega; X \rangle^*, s_i \in S$. By Corollary 3.5, we have $\langle f, g \rangle_w = \sum \beta_t \pi_t|_{s_t}$, where each $\beta_t \in k, \pi_t|_{\overline{s_t}} <_{Dl} w$. Therefore, by Lemma 3.7, we can obtain that $(f, g)_w \equiv_{ass} 0 \mod(S, w)$. Thus, S is a Gröbner-Shirshov basis in $k\langle \Omega; X \rangle$.

(ii) \Rightarrow (i). Assume that S is a Gröbner-Shirshov basis in $k\langle \Omega; X \rangle$. Then, for any composition $\langle f, g \rangle_w$ in S , we have $\langle f, g \rangle_w \in Lie(\Omega; X)$ and $\overline{\langle f, g \rangle_w} \in Id_{ass}(S)$. By Composition-Diamond lemma for associative Ω -algebras, $\langle f, g \rangle_w = \pi_1|_{\overline{s_1}} \in ALSW(\Omega; X)$. Let

$$h_1 = \langle f, g \rangle_w - \alpha_1 [\pi_1|_{s_1}]_{\overline{s_1}},$$

where α_1 is the coefficient of $\overline{\langle f, g \rangle_w}$. Then, $\overline{h_1} <_{Dl} \overline{\langle f, g \rangle_w}$, $h_1 \in Id_{ass}(S)$ and $h_1 \in Lie(\Omega; X)$. Now, the result follows from induction on $\overline{\langle f, g \rangle_w}$. \square

Lemma 3.9 *Let $S \subset Lie(\Omega; X)$ be monic and*

$$Irr(S) := \{[w]|w \in ALSW(\Omega; X), w \neq \pi|_{\overline{s}}, s \in S, \pi \in \langle \Omega; X \rangle^*\}.$$

Then, for any $h \in Lie(\Omega; X)$, h can be expressed as

$$h = \sum \alpha_i [u_i] + \sum \beta_j [\pi_j|_{s_j}]_{\overline{s_j}},$$

where each $u_i \in ALSW(\Omega; X), u_i \leq_{Dl} \bar{h}$ and $s_i \in S, \pi_j|_{\overline{s_j}} \leq_{Dl} \bar{h}$.

Proof. Since $f \in Lie(\Omega; X)$, $h = \sum \alpha_i [u_i]$, where each $u_i \in ALSW(\Omega; X)$ and $u_i >_{Dl} u_{i+1}$. If $[u_1] \in Irr(S)$, then let $h_1 = h - \alpha_1 [u_1]$. Otherwise, there exists $s_1 \in S$ such that $u_1 = \pi|_{\overline{s_1}}$. Let $h_1 = h - [\pi|_{s_1}]_{\overline{s_1}}$. In both of the above two cases, we have $h_1 \in Lie(\Omega; X)$ and $h_1 <_{Dl} \bar{h}$. Then, by induction on \bar{h} , we can obtain the result. \square

The following theorem is Composition-Diamond lemma for Lie Ω -algebras. It is an analogue of Shirshov's Composition lemma for Lie algebras [53], which was specialized to associative algebras by L. A. Bokut [9], see also G.M. Bergman [7] and B. Buchberger [19, 21].

Theorem 3.10 *(Composition-Diamond lemma for Lie Ω -algebras) Let $S \subseteq Lie(\Omega; X)$ be a non-empty monic set and $Id_{Lie}(S)$ the ideal of $Lie(\Omega; X)$ generated by S . Then the following statements are equivalent:*

(i) S is a Gröbner-Shirshov basis in $Lie(\Omega; X)$.

(ii) $f \in Id_{Lie}(S) \Rightarrow \bar{f} = \pi|_{\overline{s}} \in ALSW(\Omega; X)$ for some $s \in S$ and $\pi \in \langle \Omega; X \rangle^*$.

(iii) The set

$$Irr(S) = \{[w]|w \in ALSW(\Omega; X), w \neq \pi|_{\overline{s}}, s \in S, \pi \in \langle \Omega; X \rangle^*\}$$

is a linear basis of the Lie Ω -algebra $Lie(\Omega; X|S) := Lie(\Omega; X)/Id_{Lie}(S)$.

Proof. (i) \Rightarrow (ii) Since $f \in Id_{Lie}(S) \subseteq Id_{ass}(S)$, by Lemmas 3.8 and 3.1, we have $f = \pi|_{\bar{s}} \in ALSW(\Omega; X)$ for some $s \in S$ and $\pi \in \langle \Omega; X \rangle^*$.

(ii) \Rightarrow (iii) Suppose that $\sum \alpha_i [u_i] = 0$ in $Lie(\Omega; X|S)$, where each $[u_i] \in Irr(S)$ and $u_i >_{Dl} u_{i+1}$. That is, $\sum \alpha_i [u_i] \in Id_{Lie}(S)$. Then each α_i must be 0. Otherwise, say $\alpha_1 \neq 0$, since $\sum \alpha_i [u_i] = u_1$ and by (ii), we have $u_1 \notin Irr(S)$, a contradiction. Therefore, $Irr(S)$ is linear independent. By Lemma 3.9, $Irr(S)$ is a linear basis of $Lie(\Omega; X|S)$.

(iii) \Rightarrow (i) For any composition $\langle f, g \rangle_w$ of $f, g \in S$, we have $\langle f, g \rangle_w \in Id_{Lie}(S)$. Then, by (iii) and Lemma 3.9,

$$\langle f, g \rangle_w = \sum \beta_j [\pi_j|_{s_j}]_{\overline{s_j}},$$

where each $\beta_j \in k, \pi_j \in \langle \Omega; X \rangle^*, s_j \in S, \pi_j|_{\overline{s_j}} <_{Dl} w$. This proves that S is a Gröbner-Shirshov basis in $Lie(\Omega; X)$. \square

4 Applications

In this section, as applications of Theorem 3.10, we give Gröbner-Shirshov bases for free λ -Rota-Baxter Lie algebras, free modified λ -Rota-Baxter Lie algebras and free Nijenhuis Lie algebras and then linear bases of such three free algebras are obtained.

4.1 Free λ -Rota-Baxter Lie algebras

A Lie algebra L equipped with a linear map $P : L \rightarrow L$ satisfying

$$[P(x)P(y)] = P([P(x)y]) + P([xP(y)]) + \lambda P([xy]), \quad x, y \in L$$

is called a Rota-Baxter Lie algebra of weight λ or λ -Rota-Baxter Lie algebra. It is easy to see that any λ -Rota-Baxter Lie algebra (L, P) is a Lie Ω -algebra, where $\Omega = \{P\}$.

Let $Lie(\{P\}; X)$ be the free Lie $\{P\}$ -algebra on a set X . Denote S the set consisting of the following Lie $\{P\}$ -polynomials in $Lie(\{P\}; X)$:

$$f_{u,v} = (P([u])P([v])) - P((P([u])[v])) - P([u]P([v])) - \lambda P([u][v]),$$

where $u, v \in ALSW(\{P\}; X)$ and $u >_{Dl} v$.

It is easy to see

$$RBL(X) := Lie(\{P\}; X|S) = Lie(\{P\}; X)/Id_{Lie}(S)$$

is the free λ -Rota-Baxter Lie algebra on the set X .

Theorem 4.1 *With the order $>_{Dl}$ on $\langle \{P\}; X \rangle$, the set S is a Gröbner-Shirshov basis in $Lie(\{P\}; X)$. It follows that the set*

$$Irr(S) = \left\{ [w] \in NLSW(\{P\}; X) \left| \begin{array}{l} w \neq \pi|_{P(u)P(v)}, \quad \pi \in \langle \{P\}; X \rangle^* \\ u, v \in ALSW(\{P\}; X), u >_{Dl} v \end{array} \right. \right\}$$

is a linear basis of the free λ -Rota-Baxter Lie algebra $RBL(X) = Lie(\{P\}; X|S)$ on X .

Proof. All the possible compositions of Lie $\{P\}$ -polynomials in S are listed as below:

$$\begin{aligned} \langle f_{u,v}, f_{v,w} \rangle_{w_1}, w_1 &= P(u)P(v)P(w), u >_{Dl} v >_{Dl} w, \\ \langle f_{\pi|_{P(u)P(v)}, w}, f_{u,v} \rangle_{w_2}, w_2 &= P(\pi|_{P(u)P(v)})P(w), u >_{Dl} v, \pi|_{P(u)P(v)} >_{Dl} w, \\ \langle f_{u, \pi|_{P(v)P(w)}}, f_{v,w} \rangle_{w_3}, w_3 &= P(u)P(\pi|_{P(v)P(w)}), v >_{Dl} w, u >_{Dl} \pi|_{P(v)P(w)}. \end{aligned}$$

We check that all the compositions are trivial.

$$\begin{aligned} &\langle f_{u,v}, f_{v,w} \rangle_{w_1} \\ &= [f_{u,v}P(w)]_{\overline{f_{u,v}}} - [P(u)f_{v,w}]_{\overline{f_{v,w}}} \\ &= ((P([u])P([w]))P([v])) - (P((P([u])[v]))P([w])) - (P((([u]P([v])))P([w]))) \\ &\quad - \lambda(P((([u])[v]))P([w])) + (P([u])P((P([v])[w]))) + (P([u])P((([v]P([w]))))) \\ &\quad + \lambda(P([u])P((([v])[w])))). \end{aligned}$$

By direct computation, we have $mod(S, w_1)$:

$$\begin{aligned} &((P([u])P([w]))P([v])) \\ \equiv & (P((P([u])[w])) + P((([u]P([w]))) + \lambda P((([u])[w]))P([v])) \\ \equiv & P((P((P([u])[w]))[v])) + P(((P([u])[w])P([v]))) + \lambda P((P([u])[w])[v])) \\ &+ P((P((([u]P([w])))[v])) + P(((([u]P([w]))P([v]))) + \lambda P(((([u]P([w]))[v]))) \\ &+ \lambda P((P((([u])[w]))[v])) + \lambda P(((([u])[w])P([v]))) + \lambda^2 P(((([u])[w])[v])), \end{aligned}$$

$$\begin{aligned} &(P((([u]P([v])))P([w])) \\ \equiv & P((P((([u]P([v]))[w])) + P(((([u]P([v]))P([w]))) + \lambda P(((([u]P([v]))[w]))) \\ \equiv & P((P((([u]P([v]))[w])) + P(((([u]P([w]))P([v]))) + P((([u](P([v])P([w]))) \\ &+ \lambda P(((([u]P([v]))[w]))) \\ \equiv & P((P((([u]P([v]))[w])) + P(((([u]P([w]))P([v]))) + P((([u]P((P([v])[w]))) \\ &+ P((([u]P((([v]P([w]))))) + \lambda P((([u]P((([v])[w]))) + \lambda P(((([u]P([v]))[w])), \end{aligned}$$

$$\begin{aligned} &(P((P([u])[v]))P([w])) \\ \equiv & P((P((P([u])[v]))[w])) + P(((P([u])[v])P([w]))) + \lambda P(((P([u])[v])[w]))) \\ \equiv & P((P((P([u])[v]))[w])) + P(((P([u])P([w]))[v])) + P((P([u])((P([v])P([w]))))) \\ &+ \lambda P(((P([u])[v])[w]))) \\ \equiv & P((P((P([u])[v]))[w])) + P((P((P([u])[w]))[v])) + P((P((([u]P([w]))[v]))) \\ &+ \lambda P((P((([u])[w]))[v])) + P((P([u])((P([v])P([w]))))) + \lambda P(((P([u])[v])[w])), \end{aligned}$$

$$(P((([u])[v]))P([w])) \equiv P((P((([u])[v]))[w])) + P(((([u])[v])P([w]))) + \lambda P(((([u])[v])[w])),$$

$$\begin{aligned}
& (P([u])P((P([v])[w]))) \\
\equiv & P((P([u])(P([v])[w]))) + P([u]P((P([v])[w]))) + \lambda P([u](P([v])[w]))) \\
\equiv & P((P([u])P([v])[w])) + P((P([v])(P([u])[w]))) + P([u](P((P([v])[w]))) \\
& + \lambda P([u](P([v])[w]))) \\
\equiv & P((P((P([u])[v])[w])) + P((P([u]P([v]))[w])) + \lambda P([u](P([v])[w]))) \\
& + P((P([v])(P([u])[w]))) + P([u](P((P([v])[w]))) + \lambda P([u](P([v])[w])), \\
& (P([u])P([v]P([w]))) \\
\equiv & P((P([u])([v]P([w]))) + P([u]P([v]P([w]))) + \lambda P([u]([v]P([w])), \\
(P([u])P([v]P([w]))) \equiv & P((P([u])([v]P([w]))) + P([u]P([v]P([w]))) + \lambda P([u]([v]P([w])).
\end{aligned}$$

Therefore, we have

$$\langle f_{u,v}, f_{v,w} \rangle_{w_1} = [f_{u,v}P(w)]_{\overline{f_{u,v}}} - [P(u)f_{v,w}]_{\overline{f_{v,w}}} \equiv 0 \text{ mod}(S, w_1).$$

Denote

$$r(\pi|_{f_{u,v}}) = [\pi|_{f_{u,v}}]_{\overline{f_{u,v}}} - [\pi]_{\overline{f_{u,v}}}.$$

Then,

$$\begin{aligned}
& \langle f_{\pi|_{P(u)P(v)}, w}, f_{u,v} \rangle_{w_2} \\
= & f_{\pi|_{P(u)P(v)}, w} - [P(\pi|_{f_{u,v}})P(w)]_{P(u)P(v)} \\
= & f_{\pi|_{P(u)P(v)}, w} - (P([\pi|_{f_{u,v}}]_{P(u)P(v)})P(w)) \\
= & P((P([\pi|_{P(u)P(v)}])[w])) + P([P(u)P(v)]P([w])) + \lambda P([P(u)P(v)]P([w])) \\
& - (P(r(\pi|_{f_{u,v}}))P([w])) \\
\equiv & P((P(r(\pi|_{f_{u,v}}))[w])) + P((r(\pi|_{f_{u,v}})P([w]))) + \lambda P((r(\pi|_{f_{u,v}})[w])) \\
& - P((P(r(\pi|_{f_{u,v}}))[w])) - P((r(\pi|_{f_{u,v}})P([w]))) - \lambda P((r(\pi|_{f_{u,v}})[w])) \\
\equiv & 0 \text{ mod}(S, w_2),
\end{aligned}$$

$$\begin{aligned}
& \langle f_{u, \pi|_{P(v)P(w)}}, f_{v,w} \rangle_{w_3} \\
= & f_{u, \pi|_{P(v)P(w)}} - [P(u)P(\pi|_{f_{v,w}})]_{P(v)P(w)} \\
= & f_{u, \pi|_{P(v)P(w)}} - (P(u)P([\pi|_{f_{v,w}}]_{P(v)P(w)})) \\
= & P((P([u])[\pi|_{P(v)P(w)}])) + P([u]P([\pi|_{P(v)P(w)}])) + \lambda P([u][\pi|_{P(v)P(w)}])) \\
& - (P([u])P(r(\pi|_{f_{v,w}}))) \\
\equiv & P((P([u])r(\pi|_{f_{v,w}}))) + P([u]P(r(\pi|_{f_{v,w}}))) + \lambda P([u]r(\pi|_{f_{v,w}}))) \\
& - P((P([u])r(\pi|_{f_{v,w}}))) - P([u]P(r(\pi|_{f_{v,w}}))) - \lambda P([u]r(\pi|_{f_{v,w}}))) \\
\equiv & 0 \text{ mod}(S, w_3).
\end{aligned}$$

Therefore, S is a Gröbner-Shirshov basis in $Lie(\{P\}; X)$. By Composition-Diamond lemma for Lie Ω -algebras, the set $Irr(S)$ is a linear basis of the free λ -Rota-Baxter Lie algebra $RBL(X)$. \square

4.2 Free modified λ -Rota-Baxter Lie algebras

A Lie algebra L equipped with a linear map $P : L \rightarrow L$ satisfying

$$[P(x)P(y)] = P([P(x)y]) + P([xP(y)]) + \lambda[xy], \quad x, y \in L$$

is called a modified λ -Rota-Baxter Lie algebra. If $\lambda = -1$, the modified λ -Rota-Baxter Lie algebra is also called a Baxter Lie algebra in [20]. It is easy to see that any modified λ -Rota-Baxter Lie algebra (L, P) is a Lie Ω -algebra, where $\Omega = \{P\}$.

Let $Lie(\{P\}; X)$ be the free Lie $\{P\}$ -algebra on a set X . Denote S_B the set consisting of the following Lie $\{P\}$ -polynomials in $Lie(\{P\}; X)$:

$$f_{u,v}^B = (P([u])P([v])) - P((P([u])[v])) - P([u]P([v])) - \lambda([u][v]),$$

where $u, v \in ALSW(\{P\}; X)$ and $u >_{Dl} v$.

It is easy to see

$$MRBL(X) := Lie(\{P\}; X|S_B) = Lie(\{P\}; X)/Id_{Lie}(S_B)$$

is the free modified λ -Rota-Baxter Lie algebra on X .

Theorem 4.2 *With the order $>_{Dl}$ on $\langle \{P\}; X \rangle$, the set S_B is a Gröbner-Shirshov basis in $Lie(\{P\}; X)$. It follows that the set*

$$Irr(S_B) = \left\{ [w] \in NLSW(\{P\}; X) \mid \begin{array}{l} w \neq \pi|_{P(u)P(v)}, \quad \pi \in \langle \{P\}; X \rangle^* \\ u, v \in ALSW(\{P\}; X), u >_{Dl} v \end{array} \right\}$$

is a linear basis of the free modified λ -Rota-Baxter Lie algebra $MRBL(X) = Lie(\{P\}; X|S_B)$ on X .

Proof. All the possible compositions of Lie $\{P\}$ -polynomials in S are listed as below:

$$\begin{aligned} \langle f_{u,v}^B, f_{v,w}^B \rangle_{w_1}, w_1 &= P(u)P(v)P(w), u >_{Dl} v >_{Dl} w, \\ \langle f_{\pi|_{P(u)P(v)}, w}^B, f_{u,v}^B \rangle_{w_2}, w_2 &= P(\pi|_{P(u)P(v)})P(w), u >_{Dl} v, \pi|_{P(u)P(v)} >_{Dl} w, \\ \langle f_{u, \pi|_{P(v)P(w)}}^B, f_{v,w}^B \rangle_{w_3}, w_3 &= P(u)P(\pi|_{P(v)P(w)}), v >_{Dl} w, u >_{Dl} \pi|_{P(v)P(w)}. \end{aligned}$$

We check that all the compositions are trivial.

$$\begin{aligned} & \langle f_{u,v}^B, f_{v,w}^B \rangle_{w_1} \\ &= [f_{u,v}^B P(w)]_{\frac{f_{u,v}^B}{f_{u,v}^B}} - [P(u)f_{v,w}^B]_{\frac{f_{v,w}^B}{f_{v,w}^B}} \\ &= (((P([u])P([v])) - P((P([u])[v])) - P([u]P([v])) - \lambda([u][v]))P(w)) \\ & \quad - (P(u)((P([v])P([w])) - P((P([v])[w])) - P([v]P([w])) - \lambda([v][w])))) \\ &= ((P([u])P([w]))P([v])) - (P((P([u])[v]))P([w])) - (P([u]P([v]))P([w])) \\ & \quad - \lambda([u][v])P([w])) + (P([u])P((P([v])[w])) + (P([u])P([v]P([w])))) \\ & \quad + \lambda(P([u])([v][w])). \end{aligned}$$

By direct computation, we have $\text{mod}(S, w_1)$:

$$\begin{aligned}
& ((P([u])P([w]))P([v])) \\
\equiv & ((P((P([u])[w])) + P((P([u])P([w]))) + \lambda([u][w]))P([v])) \\
\equiv & P((P((P([u])[w]))[v])) + P((P([u])[w])P([v])) + \lambda((P([u])[w])[v]) \\
& + P((P((P([u])P([w]))[v])) + P((P([u])P([w]))P([v])) + \lambda((P([u])P([w]))[v]) \\
& + \lambda([u][w])P([v])), \\
& (P([u]P([v]))P([w])) \\
\equiv & P((P([u]P([v]))[w])) + P((P([u]P([v]))P([w]))) + \lambda([u]P([v])[w]), \\
& (P((P([u])[v]))P([w])) \\
\equiv & P((P((P([u])[v]))[w])) + P((P([u])[v])P([w])) + \lambda((P([u])[v])[w]), \\
& (P([u])P([v]P([w]))) \\
\equiv & P((P([u])(P([v])[w]))) + P([u]P([v]P([w]))) + \lambda([u](P([v])[w])), \\
& (P([u])P([v]P([w]))) \\
\equiv & P((P([u])([v]P([w]))) + P([u]P([v]P([w]))) + \lambda([u]([v]P([w])), \\
& P([u]P([v])P([w])) - P([u]P([w])P([v])) = P([u](P([v])P([w]))) \\
\equiv & P([u]P([v]P([w]))) + P([u]P([w]P([v]))) + \lambda P([u]([v]P([w])), \\
& P([u]P([v])P([w])) - P([u]P([w])P([v])) = P([u](P([v])P([w]))[v]) \\
\equiv & P([u]P([v]P([w]))[v]) + P([u]P([w]P([v]))[v]) + \lambda P([u]([v]P([w]))[v]), \\
& P([u]P([v])P([w])) - P([u]P([w])P([v])) = P([u](P([v])P([w]))[w]) \\
\equiv & P([u]P([v]P([w]))[w]) + P([u]P([w]P([v]))[w]) + \lambda P([u]([v]P([w]))[w]).
\end{aligned}$$

Therefore, we have

$$\langle f_{u,v}^B, f_{v,w}^B \rangle_{w_1} = [f_{u,v}^B P(w)]_{\overline{f_{u,v}^B}} - [P(u) f_{v,w}^B]_{\overline{f_{v,w}^B}} \equiv 0 \text{ mod}(S, w_1).$$

Denote

$$r(\pi|_{f_{u,v}^B}) = [\pi|_{f_{u,v}^B}]_{\overline{f_{u,v}^B}} - [\pi|_{\overline{f_{u,v}^B}}].$$

Then,

$$\begin{aligned}
& \langle f_{\pi|_{P(u)P(v)}, w}^B, f_{u,v}^B \rangle_{w_2} \\
&= f_{\pi|_{P(u)P(v)}, w}^B - [P(\pi|_{f_{u,v}^B})P(w)]_{P(u)P(v)} \\
&= f_{\pi|_{P(u)P(v)}, w}^B - (P([\pi|_{f_{u,v}^B}]_{P(u)P(v)})P(w)) \\
&= P((P([\pi|_{P(u)P(v)}])[w])) + P(([\pi|_{P(u)P(v)}]P([w]))) + \lambda([\pi|_{P(u)P(v)}][w]) \\
&\quad - (P(r(\pi|_{f_{u,v}^B}))P([w])) \\
&\equiv P((P(r(\pi|_{f_{u,v}^B}))[w])) + P((r(\pi|_{f_{u,v}^B})P([w]))) + \lambda(r(\pi|_{f_{u,v}^B})[w]) \\
&\quad - P((P(r(\pi|_{f_{u,v}^B}))[w])) - P((r(\pi|_{f_{u,v}^B})P([w]))) - \lambda(r(\pi|_{f_{u,v}^B})[w]) \\
&\equiv 0 \text{ mod}(S, w_2),
\end{aligned}$$

$$\begin{aligned}
& \langle f_{u, \pi|_{P(v)P(w)}}^B, f_{v,w}^B \rangle_{w_3} \\
&= f_{u, \pi|_{P(v)P(w)}}^B - [P(u)P(\pi|_{f_{v,w}^B})]_{P(v)P(w)} \\
&= f_{u, \pi|_{P(v)P(w)}}^B - (P(u)P([\pi|_{f_{v,w}^B}]_{P(v)P(w)})) \\
&= P((P([u][\pi|_{P(v)P(w)}]))) + P((([u]P([\pi|_{P(v)P(w)}]))) + \lambda([u][\pi|_{P(v)P(w)}]) \\
&\quad - (P([u])P(r(\pi|_{f_{v,w}^B})))) \\
&\equiv P((P([u])r(\pi|_{f_{v,w}^B}))) + P((([u]P(r(\pi|_{f_{v,w}^B})))) + \lambda([u]r(\pi|_{f_{v,w}^B})) \\
&\quad - P((P([u])r(\pi|_{f_{v,w}^B}))) - P((([u]P(r(\pi|_{f_{v,w}^B})))) - \lambda([u]r(\pi|_{f_{v,w}^B})) \\
&\equiv 0 \text{ mod}(S, w_3).
\end{aligned}$$

Therefore, S_B is a Gröbner-Shirshov basis in $Lie(\{P\}; X)$. By Composition-Diamond lemma for Lie Ω -algebras, the set $Irr(S_B)$ is a linear basis of the free modified λ -Rota-Baxter Lie algebra $MRBL(X)$. \square

4.3 Free Nijenhuis Lie algebras

A Lie algebra L equipped with a linear map $P : L \rightarrow L$ satisfying

$$[P(x)P(y)] = P([P(x)y]) + P([xP(y)]) - P^2([xy]), \quad x, y \in L$$

is called a Nijenhuis Lie algebra.

Let $Lie(\{P\}; X)$ be the free Lie $\{P\}$ -algebra on a set X . Denote S_N the set consisting of the following Lie $\{P\}$ -polynomials in $Lie(\{P\}; X)$:

$$f_{u,v}^N = (P([u])P([v])) - P((P([u])[v])) - P((([u]P([v]))) + P^2((([u][v])),$$

where $u, v \in ALSW(\{P\}; X)$ and $u >_{Dl} v$.

It is easy to see that

$$NL(X) := Lie(\{P\}; X|_{S_N}) = Lie(\{P\}; X)/Id_{Lie}(S_N)$$

is the free Nijenhuis Lie algebra on X .

Theorem 4.3 *With the order $>_{Dl}$ on $\langle \{P\}; X \rangle$, the set S_N is a Gröbner-Shirshov basis in $Lie(\{P\}; X)$. It follows that the set*

$$Irr(S_N) = \left\{ [w] \in NLSW(\{P\}; X) \mid \begin{array}{l} w \neq \pi|_{P(u)P(v)}, \quad \pi \in \langle \{P\}; X \rangle^* \\ u, v \in ALSW(\{P\}; X), u >_{Dl} v \end{array} \right\}$$

is a linear basis of the free Nijenhuis Lie algebra $NL(X) = Lie(\{P\}; X|S_N)$ on X .

Proof. All the possible compositions of Lie $\{P\}$ -polynomials in S are listed as below:

$$\begin{aligned} \langle f_{u,v}^N, f_{v,w}^N \rangle_{w_1}, w_1 &= P(u)P(v)P(w), u >_{Dl} v >_{Dl} w, \\ \langle f_{\pi|_{P(u)P(v)}, w}^N, f_{u,v}^N \rangle_{w_2}, w_2 &= P(\pi|_{P(u)P(v)})P(w), u >_{Dl} v, \pi|_{P(u)P(v)} >_{Dl} w, \\ \langle f_{u, \pi|_{P(v)P(w)}}^N, f_{v,w}^N \rangle_{w_3}, w_3 &= P(u)P(\pi|_{P(v)P(w)}), v >_{Dl} w, u >_{Dl} \pi|_{P(v)P(w)}. \end{aligned}$$

We check that all the compositions are trivial.

$$\begin{aligned} & \langle f_{u,v}^N, f_{v,w}^N \rangle_{w_1} \\ &= [f_{u,v}^N P(w)]_{\overline{f_{u,v}^N}} - [P(u) f_{v,w}^N]_{\overline{f_{v,w}^N}} \\ &= ((P([u])P([w]))P([v])) - (P((P([u])[v]))P([w])) - (P([u]P([v])))P([w])) \\ & \quad + (P^2([u][v]))P([w])) + (P([u])P((P([v])[w]))) + (P([u])P([v]P([w]))) \\ & \quad - (P([u])P^2([v][w]))). \end{aligned}$$

By direct computation, we have $mod(S, w_1)$:

$$\begin{aligned} & ((P([u])P([w]))P([v])) \\ \equiv & (P((P([u])[w]))P([v])) + (P([u]P([w]))P([v])) - (P^2([u][w]))P([v])) \\ \equiv & P((P((P([u])[w]))[v])) + P(((P([u])[w])P([v]))) - P^2(((P([u])[w])[v])) \\ & + P((P([u]P([w]))[v])) + P([u]P([w])P([v])) - P^2([u]P([w])[v])) \\ & - P((P^2([u][w]))[v])) - P((P([u][w])P([v]))) + P^2((P([u][w])[v])) \\ \equiv & P((P((P([u])[w]))[v])) + P(((P([u])[w])P([v]))) - P^2(((P([u])[w])[v])) \\ & + P((P([u]P([w]))[v])) + P([u]P([w])P([v])) - P^2([u]P([w])[v])) \\ & - P((P^2([u][w]))[v])) - P^2((P([u][w])[v])) - P^2([u][w]P([v])) \\ & + P^3([u][w][v])) + P^2((P([u][w])[v])), \\ & (P([u]P([v]))P([w])) \\ \equiv & P((P([u]P([v]))[w])) + P([u]P([v])P([w])) - P^2([u]P([v])[w])) \\ \equiv & P((P([u]P([v]))[w])) + P([u]P([w])P([v])) + P([u](P([v])P([w]))) \\ & - P^2([u]P([v])[w])) \\ \equiv & P((P([u]P([v]))[w])) + P([u]P([w])P([v])) + P([u]P((P([v])[w]))) \\ & + P([u]P([v]P([w]))) - P([u]P^2([v][w])) - P^2([u]P([v])[w])), \end{aligned}$$

$$\begin{aligned}
& (P((P([u])[v]))P([w])) \\
\equiv & P((P((P([u])[v]))[w])) + P(((P([u])[v])P([w]))) - P^2(((P([u])[v])[w])) \\
\equiv & P((P((P([u])[v]))[w])) + P(((P([u])P([w]))[v])) + P((P([u])([v]P([w])))) \\
& - P^2(((P([u])[v])[w])) \\
\equiv & P((P((P([u])[v]))[w])) + P((P((P([u])[w]))[v])) + P((P([u]P([w]))[v])) \\
& - P((P^2((([u][w]))[v])) + P((P([u])([v]P([w])))) - P^2(((P([u])[v])[w])),
\end{aligned}$$

$$\begin{aligned}
& (P^2((([u][v]))P([w])) \\
\equiv & P((P^2((([u][v]))[w])) + P((P((([u][v]))P([w]))) - P^2((P((([u][v]))[w])) \\
\equiv & P((P^2((([u][v]))[w])) + P^2((P((([u][v]))[w]))) + P^2((([u][v])P([w])) \\
& - P^3(((([u][v])[w])) - P^2((P((([u][v]))[w])),
\end{aligned}$$

$$\begin{aligned}
& (P([u])P((P([v])[w]))) \\
\equiv & P((P([u])(P([v])[w]))) + P((([u]P((P([v])[w]))) - P^2((([u](P([v])[w]))) \\
\equiv & P((P([u])P([v]))[w])) + P((P([v])(P([u])[w]))) + P((([u](P((P([v])[w]))) \\
& - P^2((([u](P([v])[w]))) \\
\equiv & P((P((P([u])[v]))[w])) + P((P((([u]P([v]))) [w])) - P((P^2((([u][v]))[w])) \\
& + P((P([v])(P([u])[w]))) + P((([u](P((P([v])[w]))))) - P^2((([u](P([v])[w])))),
\end{aligned}$$

$$\begin{aligned}
& (P([u])P([v]P([w]))) \\
\equiv & P((P([u])([v]P([w]))) + P((([u]P([v]P([w])))) - P^2((([u]([v]P([w])))),
\end{aligned}$$

$$\begin{aligned}
& (P([u])P^2((([v][w]))) \\
\equiv & P((P([u])P([v][w]))) + P((([u]P^2((([v][w])))) - P^2((([u]P([v][w]))) \\
\equiv & P^2((P([u])([v][w]))) + P^2((([u]P([v][w]))) - P^3((([u]([v][w]))) \\
& + P((([u]P^2((([v][w]))) - P^2((([u]P([v][w])))).
\end{aligned}$$

Therefore, we have

$$\langle f_{u,v}^N, f_{v,w}^N \rangle_{w_1} = [f_{u,v}^N P(w)]_{\overline{f_{u,v}^N}} - [P(u) f_{v,w}^N]_{\overline{f_{v,w}^N}} \equiv 0 \text{ mod}(S, w_1).$$

Denote

$$r(\pi|_{f_{u,v}^N}) = [\pi|_{f_{u,v}^N}]_{\overline{f_{u,v}^N}} - [\pi]_{\overline{f_{u,v}^N}}.$$

Then,

$$\begin{aligned}
& \langle f_{\pi|_{P(u)P(v)},w}^N, f_{u,v}^N \rangle_{w_2} \\
&= f_{\pi|_{P(u)P(v)},w}^N - [P(\pi|_{f_{u,v}^N})P(w)]_{P(u)P(v)} \\
&= f_{\pi|_{P(u)P(v)},w}^N - (P([\pi|_{f_{u,v}^N}]_{P(u)P(v)})P(w)) \\
&= P((P([\pi|_{P(u)P(v)}])[w])) + P(([\pi|_{P(u)P(v)}]P([w]))) - P^2(([\pi|_{P(u)P(v)}][w])) \\
&\quad - (P(r(\pi|_{f_{u,v}^N}))P([w])) \\
&\equiv P((P(r(\pi|_{f_{u,v}^N}))[w])) + P((r(\pi|_{f_{u,v}^N})P([w]))) - P^2((r(\pi|_{f_{u,v}^N}))[w])) \\
&\quad - P((P(r(\pi|_{f_{u,v}^N}))[w])) - P((r(\pi|_{f_{u,v}^N})P([w]))) + P^2((r(\pi|_{f_{u,v}^N}))[w])) \\
&\equiv 0 \mod(S, w_2),
\end{aligned}$$

$$\begin{aligned}
& \langle f_{u,\pi|_{P(v)P(w)}}^N, f_{v,w}^N \rangle_{w_3} \\
&= f_{u,\pi|_{P(v)P(w)}}^N - [P(u)P(\pi|_{f_{v,w}^N})]_{P(v)P(w)} \\
&= f_{u,\pi|_{P(v)P(w)}}^N - (P(u)P([\pi|_{f_{v,w}^N}]_{P(v)P(w)})) \\
&= P((P([u][\pi|_{P(v)P(w)}]))) + P((([u]P([\pi|_{P(v)P(w)}]))) - P^2((([u][\pi|_{P(v)P(w)}]))) \\
&\quad - (P([u])P(r(\pi|_{f_{v,w}^N}))) \\
&\equiv P((P([u])r(\pi|_{f_{v,w}^N}))) + P((([u]P(r(\pi|_{f_{v,w}^N})))) - P^2((([u]r(\pi|_{f_{v,w}^N})))) \\
&\quad - P((P([u])r(\pi|_{f_{v,w}^N}))) - P((([u]P(r(\pi|_{f_{v,w}^N})))) + P^2((([u]r(\pi|_{f_{v,w}^N})))) \\
&\equiv 0 \mod(S, w_3).
\end{aligned}$$

Therefore, S_N is a Gröbner-Shirshov basis in $Lie(\{P\}; X)$. By Composition-Diamond lemma for Lie Ω -algebras, the set $Irr(S_N)$ is a linear basis of the free Nijenhuis Lie algebra $NL(X)$. \square

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